Periodic systems have new classes of synchronization stability
Supplementary Material

(Dated: August 20, 2024)

I. INTRODUCTION

We present here the Master Stability Functions of several other 2-dimensional and 3-dimensional periodic systems, illustrating how they can fall into any of the 5 classes of synchronizability discussed in the conclusions of the main text.

II. THE LOTKA-VOLTERRA MODEL

The Lotka-Volterra system is a prototypical predator-prey ecological system, describing the interaction between two species [1], described by the system

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= cxy - dy \; . \end{aligned} \tag{1}$$

Choosing $a = \frac{2}{3}$, $b = \frac{4}{3}$, $c = 1$ and $d = 1$ results in a synchronous state that is always stable (Class II) for self-couplings and always unstable (Class I) otherwise (Fig. 1).

III. THE FITZHUGH–NAGUMO MODEL

The FitzHugh–Nagumo model is a simplified two-dimensional system that can model the action potential and spiking behaviour of the neuron cell as a relaxation oscillator [2, 3]. The equations defining this system are

$$\begin{aligned} \dot{x} &= x - \frac{x^3}{3} - y + I \\ c\dot{y} &= x + a - by \; . \end{aligned} \tag{2}$$

Imposing the parameter choice $I = 0.5$, $c = 12.5$, $a = 0.7$ and $b = 0.8$, the system is in Class II, with its MSF always negative for any positive coupling strength for all variable couplings except $x \to y$, for which it is in Class III (Fig. 2).

IV. THE VAN DER POL OSCILLATOR

The van der Pol oscillator, is a 2-dimensional non-conservative oscillating system with a nonlinear damping term, which, in fact, inspired the development of the FitzHugh-Nagumo model [4]. The equations that describe its dynamics are

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= a\left(1 - x^2\right) y - x \; . \end{aligned} \tag{3}$$

With the choice $a = 3.5$, we obtain a system that can be in one of three classes, depending on the coupling. Specifically, for self-couplings it is in Class II, with an always-stable synchronous state, for $x \to y$ coupling it is in Class IV, with stability only after a threshold, and for $y \to x$ coupling there is the appearance of multiple intervals of stability (Fig. 3).

V. THE CABBAGE SYSTEM

The Cabbage system is a megastable, periodically-forced oscillator with spatially-periodic damping [5]. This system has an infinite number of coexisting attractors, and its dynamics is described by the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y\cos(x) \; . \end{aligned} \tag{4}$$

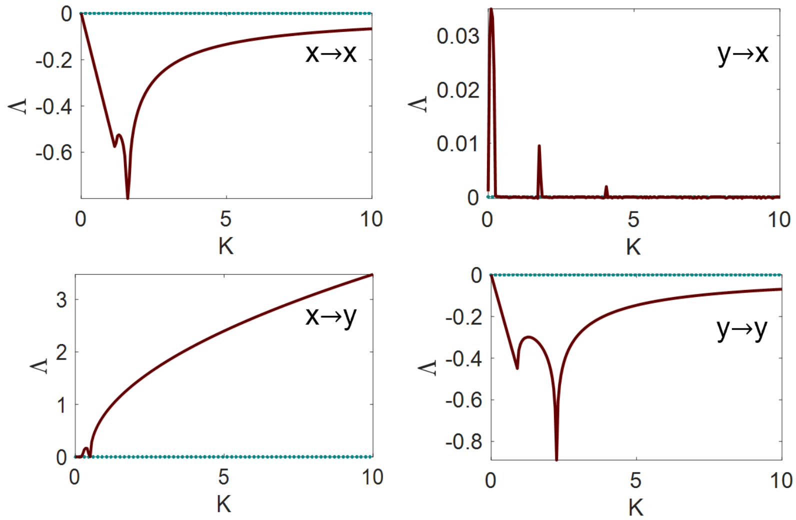

Figure 1. **Synchronization in the periodic Lotka-Volterra model can be always unstable.** The Master Stability Function (Λ) of the Lotka-Volterra model, defined in Eq. (1), as a function of the generalized coupling strength K is always positive for the $x \to y$ and $y \to x$ couplings, showing that synchronization is always unstable in these cases. The parameter values are $a = \frac{2}{3}$, $b = \frac{4}{3}$, $c = 1$ and $d = 1$.

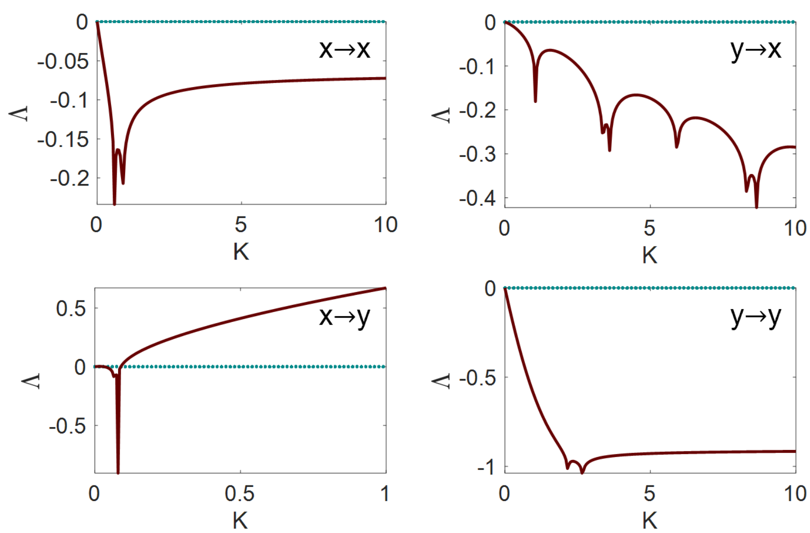

Figure 2. **Synchronizability classes of the periodic FitzHugh-Nagumo model.** The Master Stability Function (Λ) of the FitzHugh–Nagumo model, defined in Eq. (2), plotted as a function of the generalized coupling strength K, is negative within a bounded range of coupling strengths starting at 0 for the $x \to y$ coupling, and always negative otherwise. The parameter values are $I = 0.5$, $c = 12.5$, $a = 0.7$ and $b = 0.8$.

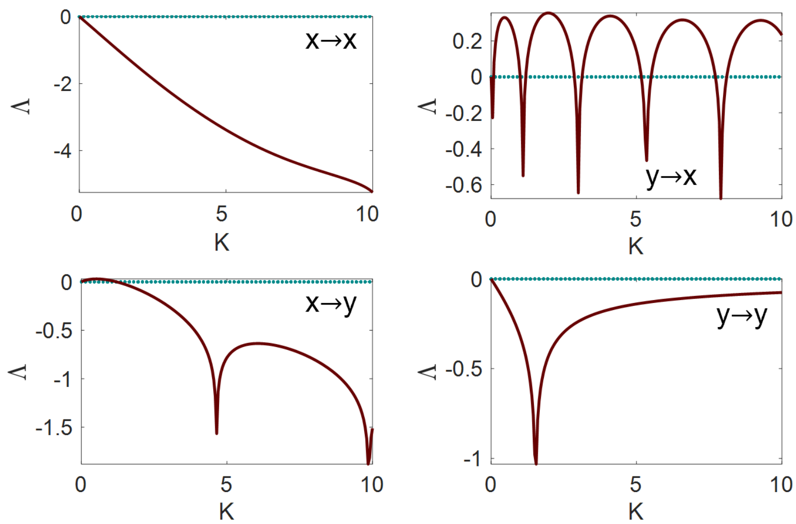

Figure 3. **New classes of synchronizability for the periodic van der Pol oscillator.** The Master Stability Function (Λ) of the van der Pol oscillator, defined in Eq. (3), plotted as a function of the generalized coupling strength K, is always negative for self-couplings, it is negative after a threshold for $x \to y$, and it has multiple ranges of coupling strength for which it is negative for $y \to x$. Note how the results for $x \to y$ and $y \to x$ couplings challenge the zero-synchronizability expectation for periodic systems.

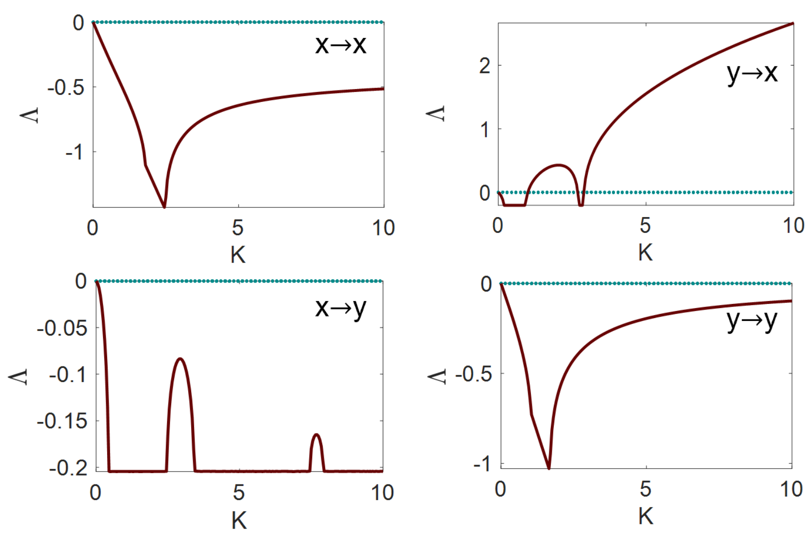

Figure 4. **New classes of synchronizability for the periodic cabbage system.** The Master Stability Function (Λ) of the cabbage system, defined in Eq. (4), plotted as a function of the generalized coupling strength K, is always negative for all couplings except $y \to x$, where it has two bounded intervals, one starting at 0, in which it is negative. This contrasts with the belief that any coupling, no matter how small will induce stable synchronization in periodic systems.

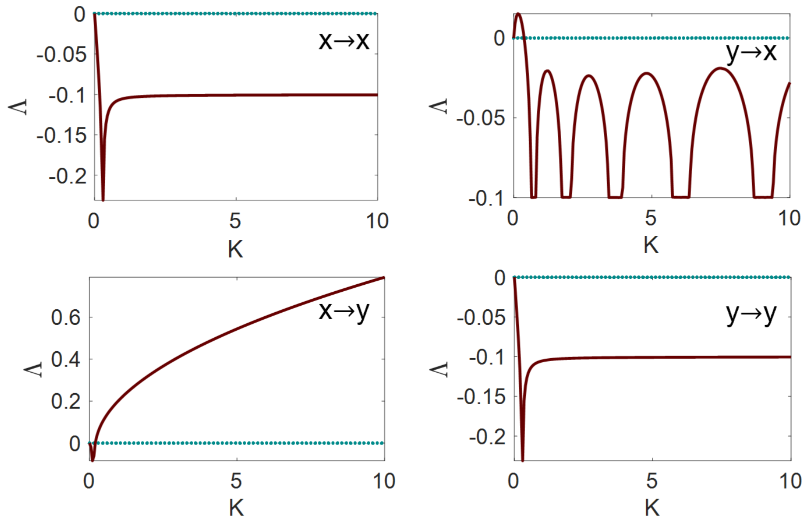

Figure 5. **New classes of synchronizability for the Stuart-Landau oscillator.** The Master Stability Function (Λ) of the Stuart-Landau oscillator, defined in Eq. (5), plotted as a function of the generalized coupling strength K, has three synchronizability classes. The MSF is always negative for all self-couplings, it is negative before a threshold for $x \to y$, and it is negative after a threshold for $y \to x$. Note that the existence of a minimum and maximum coupling strengths for stable synchronization in the last two cases is an unexpected occurrence for periodic systems. The parameter values are $\sigma_r = 0.1$, $\sigma_i = 0.2$, $l_r = 0.5$ and $l_i = 0.25$.

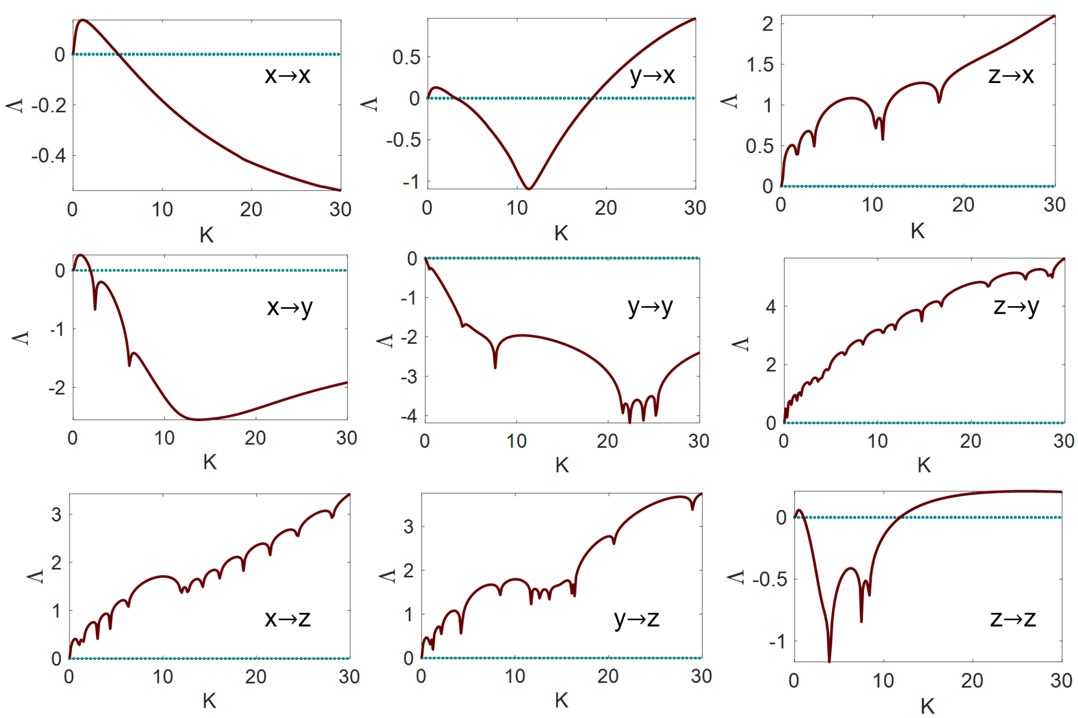

Figure 6. **New classes of synchronizability for the Lorenz system.** The Master Stability Function (Λ) of the Lorenz system, defined in Eq. (6), plotted as a function of the generalized coupling strength K, shows that it has four synchronizability classes. The MSF is always positive for $z \to x$, $z \to y$, $x \to z$ and $x \to y$ couplings, it is negative after a threshold for $x \to x$ and $x \to y$, it is negative within a bounded range of coupling strengths for $y \to x$ and $z \to z$, and it is always negative for $y \to y$. Note that all cases except $y \to y$ challenge the current beliefs about periodic systems, according to which one would expect the MSF to be always negative. The parameter values are $\sigma = 10$, $\rho = 28$ and $\beta = 0.77$.

Studying its Master Stability Function reveals that the system is always in Class II, corresponding to an always stable synchronous state, except for the coupling $y \to x$, for which there are two distinct, bounded regions of stability (Fig. 4).

VI. THE STUART-LANDAU OSCILLATOR

The Stuart–Landau oscillator [6, 7] is a well-studied system because of its fundamental relevance to Hopf bifurcations, and it has seen extensive use in the modelling of flow systems in which supercritical bifurcations occur when a control parameter exceeds a threshold. The system is defined by the equations

$$\begin{aligned} \dot{x} &= \sigma_r x - \sigma_i y - (l_r x - l_i y)\left(x^2 + y^2\right) \\ \dot{y} &= \sigma_i x + \sigma_r y - (l_i x + l_r y)\left(x^2 + y^2\right) \ . \end{aligned} \tag{5}$$

Using parameter values $\sigma_r = 0.1$, $\sigma_i = 0.2$, $l_r = 0.5$ and $l_i = 0.25$, we obtain a MSF that can belong to three different classes. Specifically, it is in Class II for self-couplings, in Class III for $x \to y$ and in Class IV for $y \to x$ (Fig. 5).

VII. THE LORENZ SYSTEM

The Lorenz system was first proposed for modeling and analyzing the seemingly unpredictable behaviour of weather [8]. Later, it found wide applications in modeling different systems, including lasers, batteries, and economic processes. The governing equations of the Lorenz system are

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= x\,(\rho - z) - y \\ \dot{z} &= xy - \beta z \ . \end{aligned} \tag{6}$$

Choosing $\sigma = 10$, $\rho = 28$ and $\beta = 0.77$, the behaviour of the MSF identifies four possible different classes for the system. More in detail, for $z \to x$, $z \to y$, $x \to z$ and $x \to y$ couplings, the system is in Class I, and synchronization is never stable. For $x \to x$ and $x \to y$, the system is in Class IV, with stable synchronization occurring only after a threshold of coupling strength. For $y \to x$ and $z \to z$, the system is in Class V, featuring a bounded region in which the synchronous state is stable. Finally, for $y \to y$, the MSF is always negative, and synchronization is always stable (Fig. 6).

[1] A. J. Lotka, P. Natl. Acad. Sci. USA **6**, 410 (1920).
[2] R. FitzHugh, Biophys. J. **1**, 445 (1961).
[3] J. Nagumo, S. Arimoto and S. Yoshizawa, P. IRE **50**, 2061 (1962).
[4] B. van der Pol, Radio Review **1**, 701 (1920).
[5] J. C. Sprott, S. Jafari, A. J. M. Khalaf and T. Kapitaniak, Eur. Phys. J. – Spec. Top. **226**, 1979 (2017).
[6] L. D. Landau, Dokl. Akad. Nauk. SSSR **44**, 311 (1944).
[7] J. T. Stuart, J. Fluid Mech. **9**, 353 (1960).
[8] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).